Chapter 5 Review Solutions

Fundamental Identities

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

Important Integration Formulas

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq 1)$$

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \sec^2 u \, du = \tan u + C$$

$$\int \csc^2 u \, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

Important Theorems and Definitions

- Comparison Theorem: If f and g are integrable and $g(x) \le f(x)$ for x in $[a,b]$, then $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$

- F is called an antiderivative of f if $F'(x) = f(x)$

- Fundamental Theorem of Calculus, Part I: Assume f is continuous on $[a,b]$. If F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- Fundamental Theorem of Calculus, Part II: Assume that f is continuous on an open interval I and let a be a point in I. Then, the area function $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of f on I; that is $A'(x) = f(x)$.

Equivalently, $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

- Consider the function $G(x) = \int_a^{g(x)} f(t)\,dt$.

Then, $G'(x) = f(g(x)) \cdot g'(x)$.

- The net change in a quantity $s(t)$ is equal to the integral of its rate of change:

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} s'(t)\,dt$$

- For an object traveling in a straight line at velocity $v(t)$,

Displacement during $[t_1, t_2]$ $= \int_{t_1}^{t_2} v(t)\,dt$

Total distance traved during $[t_1, t_2]$ $= \int_{t_1}^{t_2} |v(t)|\,dt$

Express the limit as an integral, evaluate

1. $\lim\limits_{N \to \infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right)$

Solution: Let $f(x) = \sin x$ and N a positive integer. A uniform partition of $[\pi/3, \pi/2]$ with N subintervals has

$$\Delta x = \frac{\pi}{6N} \quad \text{and} \quad x_j = \frac{\pi}{3} + \frac{\pi j}{6N}, \quad 0 \le j \le N.$$

Then, $\frac{\pi}{6N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{3}+\frac{\pi j}{6N}\right)=\Delta x\sum_{j=1}^{N}f(x_j)=R_N;$

consequently,

$$\lim_{N\to\infty}\frac{\pi}{6N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{3}+\frac{\pi j}{6N}\right)=\int_{\pi/3}^{\pi/2}\sin x\,dx=-\cos x\Big|_{\pi/3}^{\pi/2}=0+\frac{1}{2}=\frac{1}{2}$$

2. $\lim_{N\to\infty}\frac{3}{N}\sum_{k=0}^{N-1}\left(10+\frac{3k}{N}\right).$

<u>Solution)</u> Let $f(x)=x$. A uniform partition of $[10,13]$ with N subintervals has $\Delta x=\frac{3}{N}$ and $x_j=10+\frac{3j}{N}$, $0\le j\le N$.

Then, $\frac{3}{N}\sum_{k=0}^{N-1}\left(10+\frac{3k}{N}\right)=\Delta x\sum_{j=0}^{N-1}f(x_j)=L_N;$

consequently,

$$\lim_{N\to\infty}\frac{3}{N}\sum_{k=0}^{N-1}\left(10+\frac{3k}{N}\right)=\int_{10}^{13}x\,dx=\frac{1}{2}x^2\Big|_{10}^{13}=\frac{169}{2}-\frac{100}{2}=\frac{69}{2}.$$

Calculate the definate or indefinite integral

1. $\int (y+2)^4 \, dy = \frac{1}{5}(y+2)^5 + C.$

2. $\int (9t^{-2/3} + 4t^{7/3}) \, dt = 27t^{1/3} + \frac{6}{5}t^{10/3} + C.$

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3. $\int_{-2}^{4} |x-1||x-3| \, dx = \int_{-2}^{1} (x^2-4x+3) \, dx + \int_{1}^{3} (-x^2+4x-3) dx$

$+ \int_{3}^{4} (x^2-4x+3) \, dx = \frac{62}{3}.$

4. $\int t^2\sqrt{t+8} \, dt =$

Let $u = t+8$. Then, $du = dt$, $t = u-8$, and

$\int t^2\sqrt{t+8} \, dt = \int (u-8)^2\sqrt{u} \, du = \int u^{5/2} - 16u^{3/2} + 64u^{1/2} du$

$= \frac{2}{7}u^{7/2} - \frac{32}{5}u^{5/2} + \frac{128}{3}u^{3/2} + C$

$= \boxed{\frac{2}{7}(t+8)^{7/2} - \frac{32}{5}(t+8)^{5/2} + \frac{128}{3}(t+8)^{3/2} + C}$

5. $\displaystyle\int \frac{\sec^2 t}{(\tan t - 1)^2}\, dt$

Let $u = \tan t - 1$. Then, $du = \sec^2 t\, dt$. So,

$$\int \frac{\sec^2 t}{(\tan t - 1)^2} = \int u^{-2}\, du = -u^{-1} + C = \boxed{-\frac{1}{\tan t - 1} + C.}$$

6. $\displaystyle\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\, d\theta$

Let $u = \cos\theta$; then $du = -\sin\theta\, d\theta$, and the new bounds of integration are $\cos(0) = 1$ to $\cos(\pi/2) = 0$. Thus,

$$\int_0^{\pi/2} \sec^2(\cos\theta)\sin\theta\, d\theta = -\int_1^0 \sec^2 u\, du = \tan u\Big|_0^1 = \boxed{\tan 1}.$$

7. $\displaystyle\int \csc^2(9 - 2\theta)\, d\theta$

Let $u = 9 - 2\theta$. Then, $du = -2\, d\theta$ and

$$\int \csc^2(9-2\theta)\, d\theta = -\frac{1}{2}\int \csc^2 u\, du = \frac{1}{2}\cot u + C = \boxed{\frac{1}{2}\cot(9-2\theta) + C}$$

8. $\displaystyle\int \frac{(x^2+1)}{(x^3+3x)^4}\,dx$

Let $u = x^3+3x$. Then, $du = (3x^2+3)\,dx = 3(x^2+1)\,dx$ and

$$\int \frac{(x^2+1)}{(x^3+3x)^4}\,dx = \frac{1}{3}\int u^{-4}\,du = -\frac{1}{9}u^{-3} + C = \boxed{-\frac{1}{9}(x^3+3x)^{-3} + C}$$

9. $\displaystyle\int_0^{\pi/6} \sin x \cos^4 x\,dx$

Let $u = \cos x$. Then, $du = -\sin x\,dx$ and the new limits of integration are $u = 1$ and $u = \sqrt{3}/2$. Thus,

$$\int_0^{\pi/6} \sin x \cos^4 x\,dx = -\int_1^{\sqrt{3}/2} u^4\,du = -\frac{1}{5}\Big|_1^{\sqrt{3}/2} = \boxed{\frac{1}{5}\left(1 - \frac{9\sqrt{3}}{32}\right)}$$

Solve the differential equation with the given initial condition

1. $\dfrac{dy}{dx} = \sec^2 x$, $y(\pi/4) = 2$.

So, $y(x) = \int \sec^2 x\,dx = \tan x + C$., Using the condition $y(\pi/4)=2$

we find $y(\pi/4) = \tan(\pi/4) + C = 2$. So, $C = 1$.

Thus, $\boxed{y(x) = \tan x + 1}$.

2. Find $f(t)$ if $f''(t) = 1-2t$, $f(0) = 2$ and $f'(0) = -1$.

We have,
$$f'(t) = \int f''(t)dt = \int 1-2t\,dt = t - t^2 + C.$$

Using the initial condition $f'(0) = -1$, we find

$f'(0) = 0 - 0^2 + C = -1$. So, $C = -1$. Thus,

$f'(t) = t - t^2 - 1$. Now,

$$f(t) = \int f'(t)dt = \int t - t^2 - 1\,dt = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + C.$$

Using the initial condition $f(0) = 2$, we find $C = 2$.

Thus,

$$\boxed{f(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 - t + 2.}$$

3. At time $t=0$, a driver begins decelerating at a constant rate of $-10\,m/s^2$ and comes to a halt after traveling 500m. Find the velocity at $t=0$.

From the constant deceleration of $-10\,m/s$ we can determine

$$v(t) = \int -10\,dt = -10t + v_0,$$ where v_0

is the velocity of the automobile at $t=0$. Note that the automobile comes to a halt when $v(t)=0$, which occurs at

$$t = \frac{v_0}{10}\text{ s}.$$

The distance traved during the braking process is

$$s(t) = \int v(t)\,dt = -5t^2 + v_0 t + C.$$ We are given

that the braking distance is 500 meters, so

$$s\left(\frac{v_0}{10}\right) - s(0) = -5\left(\frac{v_0}{10}\right)^2 + v_0\left(\frac{v_0}{10}\right) + C - C = 500.$$

Solving for v_0 gives us $v_0 = 100\,m/s$.

Additional Problems

1. Find the local minima, the local maxima, and the points of inflection of $A(x) = \int_3^x \frac{t}{t^2+1}\,dt$.

We have, $A'(x) = \frac{x}{x^2+1}$ and

$$A''(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Now, $x=0$ is the only critical point of A; and since $A''(0) > 0$, it follows that A has a local min. at $x=0$. There are no local maxima. Moreover, A is concave down for $|x| > 1$ and concave up for $|x| < 1$. A therefore has inflection points at $x = \pm 1$.

2. With a consumption rate of $r(t) = 100 + 72t - 3t^2$, the daily consumption of water is

$$\int_0^{24} 100 + 72t - 3t^2\,dt = 100t + 36t^2 - t^3 \Big|_0^{24} = 9312 \text{ thousands of gallons}$$

From 6PM to midnight, the water consumption is

$$\int_{18}^{24} 100 + 72t - 3t^2 \, dt = 100t + 36t^2 - t^3 \Big|_{18}^{24}$$

$$= 1680 \text{ thousands of gallons.}$$

3. Evaluate $\displaystyle\int_{-8}^{8} \frac{x^{15}}{3+\cos^2 x} \, dx.$

Let $f(x) = \dfrac{x^{15}}{3+\cos^2 x}$ note that

$$f(-x) = \frac{(-x)^{15}}{3+\cos^2(-x)} = -\frac{x^{15}}{3+\cos^2 x} = -f(x).$$

So, f is odd and the interval $-8 \le x \le 8$ is symmetric about $x=0$, it follows that

$$\int_{-8}^{8} \frac{x^{15}}{3+\cos^2 x} \, dx = 0.$$

4. Find $G'(x)$ where

$$G(x) = \int_{-2}^{\sin x} t^3\,dt.$$

We have $G'(x) = \sin^3 x \cdot \frac{d}{dx}\sin x$

$= \sin^3 x \cdot \cos x.$

Find $G'(x)$ and $G'(2)$ where $G(x) = \int_0^{x^3} \sqrt{t+1}\,dt$.

$G'(x) = \sqrt{x^3+1}\,\frac{d}{dx}x^3 = 3x^2\sqrt{x^3+1}$ and so,

$G'(2) = 3(2)^2\sqrt{8+1} = 36.$

5. Use the comparison test to prove that

$$2 \le \int_1^2 2^x\,dx \le 4.$$

The function $f(x) = 2^x$ is increasing, so $1 \le x \le 2$ implies $2 = 2^1 \le 2^x \le 2^2 = 4$. So, $2 = \int_1^2 2\,dx \le \int_1^2 2^x\,dx \le \int_1^2 4\,dx = 4.$ ✓